Remarks on a note by H. Exton (on the reducibility of the Voigt functions)

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COMMENT

Remarks on a note by H Exton

Henry E Fettis
1885 California, #62, Mountain View, CA 94041, USA

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Abstract. Correct expressions for integrals given in a recent letter by Exton are found.

In a recent letter in this journal, Exton (1981) obtained the following expressions for the integrals

$$K(x, y) = \pi^{-1/2} \int_0^\infty \exp(-yr - \tfrac{1}{4}r^2) \cos(xr)\, dr \tag{1}$$

and

$$L(x, y) = \pi^{-1/2} \int_0^\infty \exp(-yr - \tfrac{1}{4}r^2) \sin(xr)\, dr \tag{2}$$

in terms of the confluent hypergeometric function ${}_1F_1(a; b; z)$:

$$K(x, y) = \exp(y^2 - x^2) \cos(2xy) + \pi^{-1/2}\{(x - iy)\,{}_1F_1[1; \tfrac{3}{2}; -(x - iy)^2] + (x + iy)\,{}_1F_1[1; \tfrac{3}{2}; -(x + iy)^2]\}, \tag{21}$$

$$L(x, y) = x\pi^{-1/2}\{{}_1F_1[1; \tfrac{1}{2}; -(x - iy)^2] + {}_1F_1[1; \tfrac{1}{2}; -(x + iy)^2]\} - \exp(y^2 - x^2) \sin(2xy). \tag{22}$$

That these last results are incorrect can be easily verified by considering the special cases $x = 0$ or $y = 0$. For example, for $x = 0$, equation (21) would give

$$K(0, y) = \exp(y^2) + \pi^{-1/2}\{-iy\,{}_1F_1(1; \tfrac{3}{2}; y^2) + iy\,{}_1F_1(1; \tfrac{3}{2}; y^2)\} = \exp(y^2),$$

whereas, from the original expression, equation (1),

$$K(0, y) = 2\pi^{-1/2} \int_0^\infty \exp(-t^2 - 2yt)\, dt = \exp(y^2)[1 - \operatorname{erf}(y)].$$

Correct expressions for these integrals can be obtained in terms of the complex error function by writing

$$\begin{aligned} K(x, y) + iL(x, y) &= (2/\sqrt{\pi}) \int_0^\infty \exp\{-[t^2 + 2t(y - ix)]\}\, dt \\ &= (2/\sqrt{\pi}) \exp(z^2) \int_z^\infty \exp(-u^2)\, du \\ &= (2/\sqrt{\pi}) \exp(z^2)[1 - \operatorname{erf}(z)] \end{aligned}$$

where $z = y - \mathrm{i}x = -\mathrm{i}(x + \mathrm{i}y)$†. By writing the error function in terms of the confluent hypergeometric function with the aid of the relation (Abramowitz and Stegun (1965), equation (7.1.21))

$$\mathrm{erf}(z) = (2z/\sqrt{\pi}) \exp(-z^2)\,_1F_1(1; \tfrac{3}{2}; z^2),$$

this becomes

$$K(x, y) + \mathrm{i}L(x, y) = \exp(z^2) - (2z/\sqrt{\pi})\,_1F_1(1; \tfrac{3}{2}; z^2).$$

Separating real and imaginary parts gives

$$K(x, y) = \exp(-x^2 + y^2)\cos(2xy) + (\mathrm{i}/\sqrt{\pi})\{(x + \mathrm{i}y)\,_1F_1[1; \tfrac{3}{2}; -(x + \mathrm{i}y)^2] - (x - \mathrm{i}y)\,_1F_1[1; \tfrac{3}{2}; -(x - \mathrm{i}y)^2]\},$$

$$L(x, y) = -\exp(-x^2 + y^2)\sin(2xy) + (1/\sqrt{\pi})\{(x + \mathrm{i}y)\,_1F_1[1; \tfrac{3}{2}; -(x + \mathrm{i}y)^2] + (x - \mathrm{i}y)\,_1F_1[1; \tfrac{3}{2}; -(x - \mathrm{i}y)^2]\}.$$

The first of the above relations would agree with equation (21) if the quantity 'i' were inserted before the brackets, and the sign between the terms within the brackets changed from '+' to '−', but since

$$_1F_1(1; \tfrac{1}{2}; z^2) = 1 + 2z^2\,_1F_1(1; \tfrac{3}{2}; z^2),$$

there does not seem to be any simple way to reconcile the second with equation (22).

Note added in proof. The author's attention has been called to similar comments by Jacob Katriel (1982 *J. Phys. A: Math. Gen.* **15** 709–10).

References

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Fried B D and Conte S D 1961 *The Plasma Dispersion Function* (New York: Academic)

† The function $K(x, y) + \mathrm{i}L(x, y)$ is, to a numerical factor, identical to the so-called 'plasma dispersion function', tabulated by Fried and Conte (1961) and by Fettis *et al* (1972).